Riemann zeta recurrence relation

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A recurrence relation representation of the Riemann zeta function

Lemma

For $\Re(s) > 1$:

$$\zeta(s) = 1 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{p_n^s} \cdot S_{p_{n-1}}(0, m, s)$$

Where p_n is the n-th prime, and $S_{p_n}(a, m, s)$ is defined by the recurrence relation:

$$\begin{aligned} S_{p_0}(a, m, s) &= \frac{1}{(1 + a + m)^s} \\ S_{p_n}(a, m, s) &= \left(\sum_{k=0}^{p_n - 1} S_{p_{n-1}}(a + k \cdot p_{n-1}\#, p_n \cdot m, s) \right) - \frac{1}{p_n^s} \cdot S_{p_{n-1}}(\frac{a}{p_n}, m, s) \end{aligned}$$

Where p_0 is the zeroth prime (initial condition), and $p_n\#$ is the n-th primorial.

Proof

Consider $\zeta(s)$ in Dirichlet series form, convergent for $\Re(s) > 1$.

$$\begin{aligned} \zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots \end{aligned}$$

Group the terms by least prime factor.

$$\begin{aligned}\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{(2\cdot 2)^s} + \frac{1}{5^s} + \frac{1}{(2\cdot 3)^s} + \frac{1}{7^s} + \cdots \\ &= 1 + \left(\frac{1}{2^s} + \frac{1}{(2\cdot 2)^s} + \frac{1}{(2\cdot 3)^s} + \cdots\right) + \left(\frac{1}{3^s} + \frac{1}{(3\cdot 3)^s} + \frac{1}{(3\cdot 5)^s} + \cdots\right) + \cdots\end{aligned}$$

Group the terms by cyclic patterns of numbers coprime with the primorial.

$$\begin{aligned}\zeta(s) &= 1 + \sum_{m=0}^{\infty} \frac{1}{(2+2\cdot m)^s} + \sum_{m=0}^{\infty} \frac{1}{(3+6\cdot m)^s} + \sum_{m=0}^{\infty} \frac{1}{(5+30\cdot m)^s} + \sum_{m=0}^{\infty} \frac{1}{(25+30\cdot m)^s} + \cdots \\ &= 1 + \sum_{m=0}^{\infty} \left(\frac{1}{2^s}\cdot\frac{1}{(1+m)^s} + \frac{1}{3^s}\cdot\frac{1}{(1+2\cdot m)^s} + \frac{1}{5^s}\cdot\left(\frac{1}{(1+6\cdot m)^s} + \frac{1}{(5+6\cdot m)^s}\right) + \cdots\right)\end{aligned}$$

Construct the primorial patterns with a recurrence relation.

For $\Re(s) > 1$:

$$\zeta(s) = 1 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{p_n^s} \cdot S_{p_{n-1}}(0, m, s)$$

Where p_n is the n-th prime, and $S_{p_n}(a, m, s)$ is defined by the recurrence relation:

$$\begin{aligned}S_{p_0}(a, m, s) &= \frac{1}{(1+a+m)^s} \\ S_2(a, m, s) &= \frac{1}{(1+a+2\cdot m)^s} \\ S_3(a, m, s) &= \frac{1}{(1+a+6\cdot m)^s} + \frac{1}{(5+a+6\cdot m)^s} \\ S_{p_n}(a, m, s) &= \left(\sum_{k=0}^{p_n-1} S_{p_{n-1}}(a + k\cdot p_{n-1}\#, p_n\cdot m, s)\right) - \frac{1}{p_n^s}\cdot S_{p_{n-1}}(\frac{a}{p_n}, m, s)\end{aligned}$$

Where p_0 is the zeroth prime (initial condition), and $p_n\#$ is the n-th primorial.

□

The recurrence relation describes a primorial-based sieve. The positive terms correspond with the numbers coprime with the primorial, or *candidate primes*. The negative part is the *candidate prime eliminator*. The eliminator is the pattern of the prior prime in the sequence of least prime factors, stretched by the new prime.

By negating the signs of the even terms we obtain Dirichlet eta $\eta(s)$, which is convergent for $\Re(s) > 0$. Dirichlet $\eta(s)$ has the same non-trivial zeros as $\zeta(s)$. All the even terms are covered by $n = 1$, so we write:

$$\begin{aligned}
\eta(s) &= 1 + \sum_{m=0}^{\infty}\left(-\frac{1}{(2+2\cdot m)^s} + \sum_{n=2}^{\infty}\frac{1}{p_n^s}\cdot S_{p_{n-1}}(0,m,s)\right) \\
&= 1 - \sum_{m=1}^{\infty}\frac{1}{(2\cdot m)^s} + \sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{1}{p_n^s}\cdot S_{p_{n-1}}(0,m,s) \\
&= 1 - \frac{1}{2^s}\cdot\zeta(s) + \sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{1}{p_n^s}\cdot S_{p_{n-1}}(0,m,s)
\end{aligned}$$

How to apply the recurrence relation

Applying the recurrence relation to obtain S_2, S_3 and S_5.

$$\begin{aligned}
S_{p_0}(a,m,s) &= \frac{1}{(1+a+m)^s} \\
S_{p_n}(a,m,s) &= \left(\sum_{k=0}^{p_n-1} S_{p_{n-1}}(a+k\cdot p_{n-1}\#, p_n\cdot m, s)\right) - \frac{1}{p_n^s}\cdot S_{p_{n-1}}(\frac{a}{p_n}, m, s)
\end{aligned}$$

Solving for S_2:

$$\begin{aligned}
S_2(a,m,s) &= S_{p_0}(a, 2\cdot m) + S_{p_0}(a+p_0\#, 2\cdot m, s) - \frac{1}{2^s}\cdot S_{p_0}(\frac{a}{2}, m, s) \\
&= \frac{1}{(1+a+2\cdot m)^s} + \frac{1}{(2+a+2\cdot m)^s} - \frac{1}{(2+a+2\cdot m)^s} \\
S_2(a,m,s) &= \frac{1}{(1+a+2\cdot m)^s}
\end{aligned}$$

Solving for S_3:

$$\begin{aligned}
S_3(a,m,s) &= \left(\sum_{k=0}^{3-1} S_2(a+k\cdot 2\#, 3\cdot m, s)\right) - \frac{1}{3^s}\cdot S_2(\frac{a}{3}, m, s) \\
&= S_2(a, m\cdot 3, s) + S_2(a+2, 3\cdot m, s) + S_2(a+4, 3\cdot m, s) - \frac{1}{(3+a+6\cdot m)^s} \\
&= \frac{1}{(1+a+6\cdot m)^s} + \frac{1}{(3+a+6\cdot m)^s} + \frac{1}{(5+a+6\cdot m)^s} - \frac{1}{(3+a+6\cdot m)^s} \\
S_3(a,m,s) &= \frac{1}{(1+a+6\cdot m)^s} + \frac{1}{(5+a+6\cdot m)^s}
\end{aligned}$$

Solving for S_5:

$$\begin{aligned}
S_5(a,m,s) &= \left(\sum_{k=0}^{5-1} S_3(a+k\cdot 3\#, 5\cdot m, s)\right) - \frac{1}{5^s}\cdot S_3(\frac{a}{5}, m, s) \\
&= S_3(a, 5\cdot m, s) + S_3(a+6, 5\cdot m, s) + S_3(a+12, 5\cdot m, s) \\
&\quad + S_3(a+18, m\cdot 5, s) + S_3(a+24, 5\cdot m, s) - \frac{1}{5^s}\cdot S_3(\frac{a}{5}, m, s) \\
&= \frac{1}{(1+a+30\cdot m)^s} + \frac{1}{(5+a+30\cdot m)^s} \\
&\quad + \frac{1}{(7+a+30\cdot m)^s} + \frac{1}{(11+a+30\cdot m)^s} \\
&\quad + \frac{1}{(13+a+30\cdot m)^s} + \frac{1}{(17+a+30\cdot m)^s} \\
&\quad + \frac{1}{(19+a+30\cdot m)^s} + \frac{1}{(23+a+30\cdot m)^s} \\
&\quad + \frac{1}{(25+a+30\cdot m)^s} + \frac{1}{(29+a+30\cdot m)^s} \\
&\quad - \frac{1}{(5+a+30\cdot m)^s} - \frac{1}{(25+a+30\cdot m)^s} \\
S_5(a,m,s) &= \frac{1}{(1+a+30\cdot m)^s} + \frac{1}{(7+a+30\cdot m)^s} + \frac{1}{(11+a+30\cdot m)^s} \\
&\quad + \frac{1}{(13+a+30\cdot m)^s} + \frac{1}{(17+a+30\cdot m)^s} + \frac{1}{(19+a+30\cdot m)^s} \\
&\quad + \frac{1}{(23+a+30\cdot m)^s} + \frac{1}{(29+a+30\cdot m)^s}
\end{aligned}$$

Congruence relations and residue systems

The number of terms in S_{p_n} is analogous to the size of the set of congruence relations that any prime greater than p_n must satisfy one of.

Every prime p greater than 2 satisfies the congruence relation:

$$p \equiv 1 \pmod 2$$

Every prime p greater than 3 satisfies one of the congruence relations:

$$\begin{aligned} p &\equiv 1 \pmod 6 \\ p &\equiv 5 \pmod 6 \end{aligned}$$

Every prime p greater than 5 satisfies one of the congruence relations:

$$
\begin{aligned}
p &\equiv 1 \pmod{30} \\
p &\equiv 7 \pmod{30} \\
p &\equiv 11 \pmod{30} \\
p &\equiv 13 \pmod{30} \\
p &\equiv 17 \pmod{30} \\
p &\equiv 19 \pmod{30} \\
p &\equiv 23 \pmod{30} \\
p &\equiv 29 \pmod{30}
\end{aligned}
$$

The number of congruence relations C_{p_n} added at step n is equal to the number of coprimes with the n-th primorial (up to the n-th primorial).

$$
\begin{aligned}
C_{p_n} &= \prod_{i=1}^{n} (p_i - 1) \\
&= \phi(p_n\#)
\end{aligned}
$$

Where $p_n\#$ is the n-th primorial, and ϕ is Euler's totient function.