The Riemann zeta sieve

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Abstract

The Riemann zeta sieve is the sieve of Eratosthenes embedded in a recurrence relation that generates the Riemann zeta function.

The Riemann zeta sieve

Theorem: Consider $\zeta(s)$ in Dirichlet series form, convergent for $\Re(s) > 1$.

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots\end{aligned}$$

This series is equivalent to the series generated by the following formula.

$$\zeta(s) = 1 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{p_n^s} \cdot S_{p_{n-1}}(0, m, s) \tag{1}$$

Where p_n is the n-th prime, and $S_{p_n}(a, m, s)$ is the recurrence relation:

$$\begin{aligned}S_{p_0}(a, m, s) &= \frac{1}{(1 + a + m)^s} \\ S_{p_n}(a, m, s) &= \left(\sum_{k=0}^{p_n - 1} S_{p_{n-1}}(a + k \cdot p_{n-1}\#, p_n \cdot m, s) \right) - \frac{1}{p_n^s} \cdot S_{p_{n-1}}(\frac{a}{p_n}, m, s)\end{aligned} \tag{2}$$

Where p_0 is the zeroth prime (initial condition), and $p_n\#$ is the n-th primorial.

Proof: Consider the summation formula for $\zeta(s)$, in Dirichlet series form, convergent for $\Re(s) > 1$.

$$\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} \\ &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \frac{1}{7^s} + \cdots\end{aligned}$$

Group the terms by least prime factor.

$$\begin{aligned}\zeta(s) = 1 &+ \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right) + \left(\frac{1}{3^s} + \frac{1}{9^s} + \frac{1}{15^s} + \cdots\right) \\ &+ \left(\frac{1}{5^s} + \frac{1}{25^s} + \frac{1}{35^s} + \cdots\right)\cdots\end{aligned}$$

Replace these groups with sums that generate the cyclic patterns of numbers coprime with the primorial.

$$\begin{aligned}\zeta(s) = 1 &+ \sum_{m=0}^{\infty} \frac{1}{(2+2\cdot m)^s} + \sum_{m=0}^{\infty} \frac{1}{(3+6\cdot m)^s} \\ &+ \left(\sum_{m=0}^{\infty} \frac{1}{(5+30\cdot m)^s} + \sum_{m=0}^{\infty} \frac{1}{(25+30\cdot m)^s}\right) + \cdots\end{aligned}$$

Combine the m summations and factor out the least prime factor.

$$\zeta(s) = 1 + \sum_{m=0}^{\infty} \left(\frac{1}{2^s} \cdot \frac{1}{(1+m)^s} + \frac{1}{3^s} \cdot \frac{1}{(1+2\cdot m)^s} + \frac{1}{5^s} \cdot \left(\frac{1}{(1+6\cdot m)^s} + \frac{1}{(5+6\cdot m)^s}\right) + \cdots\right)$$

Generate these primorial patterns with sums of recurrence relations.

For $\Re(s) > 1$:

$$\zeta(s) = 1 + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \frac{1}{p_n^s} \cdot S_{p_{n-1}}(0, m, s)$$

Where p_n is the n-th prime, and $S_{p_n}(a, m, s)$ is defined by the recurrence relation:

$$
\begin{aligned}
S_{p_0}(a,m,s) &= \frac{1}{(1+a+m)^s} \\
S_2(a,m,s) &= \frac{1}{(1+a+2\cdot m)^s} \\
S_3(a,m,s) &= \frac{1}{(1+a+6\cdot m)^s} + \frac{1}{(5+a+6\cdot m)^s} \\
S_{p_n}(a,m,s) &= \left(\sum_{k=0}^{p_n-1} S_{p_{n-1}}(a+k\cdot p_{n-1}\#, p_n\cdot m, s)\right) - \frac{1}{p_n^s}\cdot S_{p_{n-1}}(\frac{a}{p_n}, m, s)
\end{aligned}
$$

Where p_0 is the zeroth prime (initial condition), and $p_n\#$ is the n-th primorial.

□

Calculating S_{p_n} is equivalent in operation to applying the sieve of Eratosthenes. The positive terms correspond with the numbers coprime with the primorial, or candidate primes. The negative part is the candidate prime eliminator. The eliminator is the pattern of the prior prime, multiplied by the new prime. Equivalently, the eliminator is the cyclic pattern of the new prime in the sequence of least prime factors, obtained by multiplying the new prime with the numbers coprime with the primorial of the prior prime.

By negating the signs of the even terms we obtain Dirichlet eta $\eta(s)$, convergent for $\Re(s) > 0$. All the even terms are covered by $n = 1$, so we write:

$$
\begin{aligned}
\eta(s) &= 1 + \sum_{m=0}^{\infty}\left(-\frac{1}{(2+2\cdot m)^s} + \sum_{n=2}^{\infty}\frac{1}{p_n^s}\cdot S_{p_{n-1}}(0,m,s)\right) \\
&= 1 - \sum_{m=1}^{\infty}\frac{1}{(2\cdot m)^s} + \sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{1}{p_n^s}\cdot S_{p_{n-1}}(0,m,s) \\
&= 1 - \frac{1}{2^s}\cdot\zeta(s) + \sum_{m=0}^{\infty}\sum_{n=2}^{\infty}\frac{1}{p_n^s}\cdot S_{p_{n-1}}(0,m,s)
\end{aligned}
$$

Calculating S_{p_n}

Applying the recurrence relation to obtain S_2, S_3 and S_5, starting with initial condition S_{p_0}.

$$
\begin{aligned}
S_{p_0}(a,m,s) &= \frac{1}{(1+a+m)^s} \\
S_{p_n}(a,m,s) &= \left(\sum_{k=0}^{p_n-1} S_{p_{n-1}}(a+k\cdot p_{n-1}\#, p_n\cdot m, s)\right) - \frac{1}{p_n^s}\cdot S_{p_{n-1}}(\frac{a}{p_n}, m, s)
\end{aligned}
$$

Solving for S_2:

$$\begin{aligned}
S_2(a,m,s) &= S_{p_0}(a, 2\cdot m) + S_{p_0}(a+p_0\#, 2\cdot m, s) - \frac{1}{2^s}\cdot S_{p_0}(\frac{a}{2}, m, s)\\
&= \frac{1}{(1+a+2\cdot m)^s} + \frac{1}{(2+a+2\cdot m)^s} - \frac{1}{(2+a+2\cdot m)^s}\\
S_2(a,m,s) &= \frac{1}{(1+a+2\cdot m)^s}
\end{aligned}$$

Solving for S_3:

$$\begin{aligned}
S_3(a,m,s) &= \left(\sum_{k=0}^{3-1} S_2(a+k\cdot 2\#, 3\cdot m, s)\right) - \frac{1}{3^s}\cdot S_2(\frac{a}{3}, m, s)\\
&= S_2(a, m\cdot 3, s) + S_2(a+2, 3\cdot m, s) + S_2(a+4, 3\cdot m, s) - \frac{1}{(3+a+6\cdot m)^s}\\
&= \frac{1}{(1+a+6\cdot m)^s} + \frac{1}{(3+a+6\cdot m)^s} + \frac{1}{(5+a+6\cdot m)^s} - \frac{1}{(3+a+6\cdot m)^s}\\
S_3(a,m,s) &= \frac{1}{(1+a+6\cdot m)^s} + \frac{1}{(5+a+6\cdot m)^s}
\end{aligned}$$

Solving for S_5:

$$\begin{aligned}
S_5(a,m,s) &= \left(\sum_{k=0}^{5-1} S_3(a+k\cdot 3\#, 5\cdot m, s)\right) - \frac{1}{5^s}\cdot S_3(\frac{a}{5}, m, s)\\
&= S_3(a, 5\cdot m, s) + S_3(a+6, 5\cdot m, s) + S_3(a+12, 5\cdot m, s)\\
&\quad + S_3(a+18, m\cdot 5, s) + S_3(a+24, 5\cdot m, s) - \frac{1}{5^s}\cdot S_3(\frac{a}{5}, m, s)\\
&= \frac{1}{(1+a+30\cdot m)^s} + \frac{1}{(5+a+30\cdot m)^s}\\
&\quad + \frac{1}{(7+a+30\cdot m)^s} + \frac{1}{(11+a+30\cdot m)^s}\\
&\quad + \frac{1}{(13+a+30\cdot m)^s} + \frac{1}{(17+a+30\cdot m)^s}\\
&\quad + \frac{1}{(19+a+30\cdot m)^s} + \frac{1}{(23+a+30\cdot m)^s}\\
&\quad + \frac{1}{(25+a+30\cdot m)^s} + \frac{1}{(29+a+30\cdot m)^s}\\
&\quad - \frac{1}{(5+a+30\cdot m)^s} - \frac{1}{(25+a+30\cdot m)^s}
\end{aligned}$$

$$
\begin{aligned}
S_5(a,m,s) = &\frac{1}{(1+a+30\cdot m)^s} + \frac{1}{(7+a+30\cdot m)^s} + \frac{1}{(11+a+30\cdot m)^s} \\
&+ \frac{1}{(13+a+30\cdot m)^s} + \frac{1}{(17+a+30\cdot m)^s} + \frac{1}{(19+a+30\cdot m)^s} \\
&+ \frac{1}{(23+a+30\cdot m)^s} + \frac{1}{(29+a+30\cdot m)^s}
\end{aligned}
$$

Congruence relations and residue systems

The number of terms in S_{p_n} is analogous to the size of the set of congruence relations that any prime greater than p_n must satisfy one of.

Every prime p greater than 2 satisfies the congruence relation:

$$p \equiv 1 \pmod 2$$

Every prime p greater than 3 satisfies one of the congruence relations:

$$
\begin{aligned}
p &\equiv 1 \pmod 6 \\
p &\equiv 5 \pmod 6
\end{aligned}
$$

Every prime p greater than 5 satisfies one of the congruence relations:

$$
\begin{aligned}
p &\equiv 1 \pmod{30} \\
p &\equiv 7 \pmod{30} \\
p &\equiv 11 \pmod{30} \\
p &\equiv 13 \pmod{30} \\
p &\equiv 17 \pmod{30} \\
p &\equiv 19 \pmod{30} \\
p &\equiv 23 \pmod{30} \\
p &\equiv 29 \pmod{30}
\end{aligned}
$$

The number of congruence relations R_{p_n} added at step n is equal to the number of coprimes with the n-th primorial, up to the n-th primorial.

$$
\begin{aligned}
R_{p_n} &= \prod_{i=1}^{n} (p_i - 1) \\
&= \phi(p_n\#)
\end{aligned}
$$

Where $p_n\#$ is the n-th primorial, and ϕ is Euler's totient function.